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**SUBSPACE CORRECTION METHODS
FOR THE ITERATIVE SOLUTION OF HIERARCHIC PLATE MODELS
I: HEAT TRANSFER PROBLEMS**

by

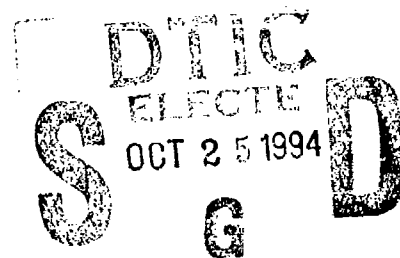**C. Schwab**

and

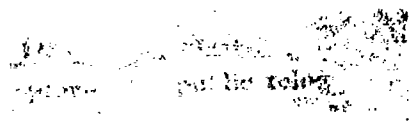**I. Babuška**

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SUBSPACE CORRECTION METHODS
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I: HEAT TRANSFER PROBLEMS

by

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Abstract

A class of iterative methods for solving the system of partial differential equations stemming from hierarchic models of the heat conduction problem in orthotropic laminated plates is analyzed. The analysis leads to sharp explicit expressions for the rate of the convergence depending on the shape or director functions used in the derivations of the hierarchic models.

INTRODUCTION

The problem of partial differential equations on thin domains is a basic problem
in engineering. We mention, as an example, the typical problem of the homogeneous or
laminated plates and shells. The usual theory is either based on physical models (see,
e.g., [1], [2], [3], [4]), the asymptotic models (see e.g., [5], [6]) or the hierarchic models
(see, e.g., [7], [8], [9], [10]) which reduce the original 3 dimensional problem to a coupled
system of partial differential equations in two dimensions. The main idea behind the
hierarchic modelling is to create a sequence of two dimensional problems the solutions of
which converge to the solution of the original three dimensional problem. The problem
of a-posteriori error estimation for this dimension reduction was addressed in [9].

The main computational problem arises from the obvious fact that the solution of
the higher order, hierarchical formulation is much more computationally expensive than
the lowest one. Hence, a major question arises, how to solve the two dimensional elliptic
system by an iteration procedure. The method proposed here falls in the general
category of subspace correction methods.

The convergence of (parallel and successive) subspace correction methods under
fairly general conditions on the space splittings is standard by now [11]. The most widely
used splittings are either related to domain decompositions or to h-multilevel methods.
Frequently, however, sharp estimates of the convergence factors cannot be obtained for

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situations of engineering interest.

In the present paper, the function space used to derive the hierarchical plate model is split with respect to the (spectral) order of the transverse shape functions used in the derivation of the hierarchical plate model. Further, a Fourier analysis of the resulting subspace correction method is performed and explicit and sharp expressions for the convergence factors of the subspace correction method in terms of the transverse shape functions "directors" are obtained. This allows in particular to (computationally) optimize the shape functions used in the dimensional reduction with respect to the convergence rate of the subspace correction method.

The outline of this paper is as follows: in Section 2 we present the heat conduction problem and the hierarchic models. In Section 3 we introduce the hierarchical subspace correction methods. Section 4 introduces the tool of Fourier analysis and contains the main results. Section 5 discusses in more details the Fourier analysis of the subspace corrections in dependence on the basis used in the modelling. Section 6 finally presents numerical experiments.

2. THE BOUNDARY VALUE PROBLEM AND THE HIERARCHY OF MODELS

For $\alpha_i \in \mathbb{R}_+^n$, we define the (hyper) rectangle

(2.1)
$$\omega_\alpha := \{x \in \mathbb{R}^n \mid |x_i| < \alpha_i/2, \ 1 \le i \le n\}.$$

We will be mainly interested in $n = 1$ or 2, but our analysis is for convenience for any

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$n \in \mathbb{N}$. With ω_{a} and $0 < d < 1$ we associate the domain

$$(2.2) \qquad \Omega := \omega_{a} \times (-d,d)$$

of thickness $t = 2d$ and the "edge"

$$(2.3) \qquad \Gamma = \gamma \times (-d,d), \quad \gamma = \partial\omega_{a}.$$

Points in Ω shall be denoted by (x,y) where $x \in \omega_{a}$, $|y| < d$. We also define the faces

$$(2.4) \qquad R_{\pm} = \{(x,y) \,|\, x \in \omega_{a}, \ y = \pm d\}.$$

In Ω we consider the boundary value problem

$$(2.5) \qquad \begin{aligned} Lu &= 0 \text{ in } \Omega \\ u &= 0 \text{ on } \Gamma, \\ Du &= f^{\pm} \text{ on } R_{\pm}, \end{aligned}$$

where D denotes the conormal derivative and L is defined (in the sense of distributions) by

$$Lu = \frac{\partial}{\partial y}\left[a\left(\frac{y}{d}\right)\frac{\partial u}{\partial y}\right] + b\left(\frac{y}{d}\right)\nabla_{x}\cdot C\nabla_{x},$$

where ∇_{x} denotes the gradient with respect to $x \in \mathbb{R}^{a}$ and $a,b \in L^{\infty}(-1,1)$ satisfy

$$(2.6) \qquad 0 < \underline{a} \leq a(z), \quad 0 < \underline{b} \leq b(z)$$

and $C = C^{T}$ is an $n \times n$ matrix for which

$$(2.7) \qquad 0 < \underline{c}\,|\xi|^{2} \leq \xi^{T}C\xi, \quad \forall\, 0 \neq \xi \in \mathbb{R}^{a}.$$

For the weak formulation of (2.5) define

$$H^1(\Omega,\Gamma) = H^1(\Omega) \cap \{u \mid u = 0 \text{ on } \Gamma\}.$$

Then (2.5) becomes: Find $u \in H^1(\Omega,\Gamma)$ such that

(2.8)
$$B(u,v) = F(v), \quad \forall v \in H^1(\Omega,\Gamma),$$

where

(2.9)
$$B(u,v) = \int_\Omega \left\{ a\left[\frac{y}{d}\right] \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} + b\left[\frac{y}{d}\right] \nabla_x v \cdot C \nabla_x u \right\} dx\,dy,$$

(2.10)
$$F(v) = \int_{\omega_a} \{f^+(x)v(x,d) + f^-(x)v(x,-d)\}\,dx.$$

Under the assumptions (2.6), (2.7) the problem (2.8) has a unique weak solution $u \in H^1(\Omega,\Gamma)$ provided that, for example $f^+, f^- \in L^2(\omega_a)$ (this could be weakened). For simplicity we assume also that

$$a(z) = a(-z), \quad b(z) = b(-z), \quad \text{a.e.} \quad z \in (-1,1)$$

and that

(2.11)
$$f^+ = f^- = f,$$

which ensures $u(x,y) = u(x,-y)$ a.e. in Ω.

Hierarchical models of (2.8) are approximations of (2.8) by elliptic boundary value problems on ω_a, which are obtained from (2.8) by a dimensional reduction method which we now describe. Let

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(2.12)
$$S(q) = \left\{ u = \sum_{j=0}^{q} X_j(x)\varphi_j\left(\frac{y}{d}\right), \quad X_j(x) \in H_0^1(\omega_e) \right\}$$

with linearly independent director functions $\varphi = (\varphi_0,...,\varphi_q)^T$, and such that span $\{\varphi_j\}_{j=0}^q = \text{span}\{\psi_j\}_{j=0}^q$ where $\psi_j(z) = \psi_j(-z)$ and the functions ψ_j are linearly independent and recursively defined by

(2.13)
$$\int_{-1}^{1} a(z)\psi_0' v' dz = 0,$$

(2.14)
$$\int_{-1}^{1} a(z)\psi_1' v' dz + \int_{-1}^{1} b(z)\psi_0 v dz = v(1) + v(-1),$$

(2.15)
$$\int_{-1}^{1} a(z)\psi_j' v' dz + \int_{-1}^{1} b(z)\psi_{j-1} v dz = 0, \quad j = 2,3,... \quad \forall v \in H^1(-1,1).$$

This selection of the functions ψ_j ensures certain optimality properties of the hierarchical models [7]. Moreover, it was shown in [7] that

(2.16)
$$\{\psi_j\}_{j=0}^{\infty} \text{ is dense in } H^1(-1,1) \cap \{\psi(z) = \psi(-z)\}$$

<u>Remark 2.1</u>. From (2.13) it follows that ψ_0 is constant.

The sequence of Neumann problems (2.14), (2.15) defines the functions ψ_j uniquely since the constant for ψ_j is uniquely determined by the compatibility condition ensuring the existence of ψ_{j+1}. $\quad\square$

Obviously, the space $S(q)$ in (2.12) is a closed, linear subspace of $H^1(\Omega,\Gamma)$ and hence the dimensionally reduced problem is : find $u_q \in S(q)$ such that

$$(2.17) \qquad\qquad B(u_q, v) = F(v), \quad \forall v \in S(q).$$

This problem admits a unique solution u_q. The problem (2.17) results in the following

boundary value problem for $X := (X_0, X_1, ..., X_q)^T \in [H_0^1(\omega_a)]^{q+1}$:

$$(2.18) \qquad L(D_x)X := -d^2 AP(D_x)X + BX = 2df(x)\varphi(1) \quad \text{in} \quad \omega_a$$

$$X = 0 \quad \text{on} \quad \partial\omega_a,$$

where $\varphi(z) := \{\varphi_0, \varphi_1, ..., \varphi_q\}^T$ and

$$(2.19) \qquad A = \int_{-1}^{1} b(z)\varphi\varphi^T dz, \quad B = \int_{-1}^{1} a(z)\varphi'\varphi'^T \, dz$$

and the differential operator $P(D_x)$ is given by

$$(2.20) \qquad\qquad P(D_x) = \nabla_x \cdot C\nabla_x.$$

Remark 2.2 More generally than (2.12), $S(q)$ may be defined to admit a model order

q which varies throughout ω_a. This will not be considered here and we refer to [13],

[14] for details. □

Remark 2.3. Other boundary conditions on Γ could of course also be considered. □

Remark 2.4. In general, the boundary value problems (2.18) must be solved

approximately, e.g., by the finite element method, but we will not address this

approximation explicitly and assume that all boundary value problems on ω_a are solved

exactly. □

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Remark 2.5. Define $V(q) := \text{span}\{\psi_j\}_{j=0}^q \subset H^1(-1,1)$. Then obviously $S(q) = H_0^1(\omega_*) \otimes V_q$ and the dimensionally reduced solution u_q in (2.17) is independent of the particular basis $\{\varphi_j\}_{j=0}^q$ of V_q used in the definition (2.12) of $S(q)$.

□

3.   HIERARCHICAL SUBSPACE CORRECTION METHOD (HCM)

Solving the problem (2.18) by the finite element method (cf. Remark 2.4), a large system of linear equations has to be solved. We formulate therefore *block iterative techniques* for the solution of these systems, each block corresponding to one equation in (2.18), with a size comparable to that of the simplest model in the hierarchy for $q = 0$.

We assume here that the equations (2.18) of the hierarchical model have been discretized with high accuracy. This allows to formulate and analyze the iterative scheme in the *semidiscrete setting*, i.e. under the assumption that (2.18) is solved exactly.

All methods considered can also be interpreted as *subspace correction methods* (see, for example, [11]) based on a suitable *splitting* of $S(q)$ in (2.12) which we now introduce: we write

(3.1) $$S(q) = S_0 + S_1 + \cdots + S_q$$

where

$$S_j := H_0^1(\omega_*) \otimes \left\{ \varphi_j\left[\frac{y}{d}\right] \right\}$$

and

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$$\text{span}\{\varphi_j\}_{j=0}^q \;=\; \text{span}\{\psi_j\}_{j=0}^q,$$

with ψ_j as in (2.13)-(2.15). Since the φ_j are by assumption linearly independent, we have

(3.2) $\qquad\qquad S_m \cap S_n \;=\; \{0\}, \quad m \neq n, \quad 0 \leq m, \quad n \leq q.$

We can therefore write

$$u_q \;=\; \sum_{j=0}^q U_j^q, \quad U_j^q \;=\; X_j(x)\varphi_j\left[\frac{y}{d}\right] \in S_j$$

and (2.15) takes the form

(3.3) $\qquad\qquad \sum_{j=0}^q B(U_j^q, v_k) \;=\; F(v_k), \quad \forall v_k \in S_k, \quad k = 0,\ldots,q,$

where $v \;=\; \sum_{k=0}^q v_k \in S(q), \quad v_k \in S_k.$ Evidently, (3.3) and (2.17) are equivalent weak formulations of the reduced problem (2.18).

Our subspace correction algorithms will be based on the decomposition (3.1). We begin by formulating a successive subspace correction algorithm with relaxation which becomes, upon FE discretization of (2.18) in the x variable, a block SOR-method.

Algorithm 3.1. 1. Given a relaxation parameter $0 < \theta < 2$, and an initial approximation u_q^0. Set $n = 0$.

    2. Repeat until convergence based on a given tolerance:

    2.1. $\quad u_q^n \;=\; \sum_{j=0}^q U_j^{(n)}, \quad U_j^{(n)} \in S_j$

    2.2. For $k = 0,\ldots,q$ solve the problem: Find $U_k^{(n+1)} \in S_k$ such that

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$$B\left(U_k^{(n+1)}, v_k\right) \;=\; (1-\theta) B\left(U_k^{(n)}, v_k\right)$$

$$-\; \theta \left\{ \sum_{j<k} B\left(U_j^{(n+1)}, v_k\right) + \sum_{j>k} B\left(U_j^{(n)}, v_k\right) \right\}$$

(3.4)

$$+\; \theta\, F(v_k), \quad \forall v_k \in S_k$$

2.3.   Set

$$u_q^{(n+1)} \;=\; \sum_{j=0}^{q} U_j^{(n+1)},$$

$$n := n + 1.$$

2.4.   end.

We already observed in Remark 2.5 that there is considerable freedom in the selection of the basis functions φ_j, which does not affect u_q. It is apparent, however, that the space decomposition (3.1) strongly depends on the basis $\{\varphi_j\}$ chosen in (2.12). This, in turn, shows that the convergence properties of Algorithm 3.1 are governed by the basis $\{\varphi_j\}$. Our purpose in the following sections is a quantitative analysis of the convergence rate κ of the Algorithm 3.1 depending on the basis $\{\varphi_j\}$ and the data $f(x)$ of the problem. By *convergence rate* we mean the smallest number $\kappa \in [0,1]$ such that for any $u_q^{(n)}$,

(3.5)
$$\left\| u_q - u_q^{(n+1)} \right\| \;\leq\; \kappa \left\| u_q - u_q^{(n)} \right\|_{E(\Omega)}, \quad \forall n \geq 0$$

where $\|v\|_{E(\Omega)} = (B(v,v))^{1/2}$.

The convergence rate κ in (3.5) guarantees a reduction of the error by a factor

κ per step. Frequently, however, the *asymptotic convergence rate*

$$(3.6) \qquad \rho \; = \; \sup_{f \in L^2(\omega_\alpha)} \left\{ \varlimsup_{k \to \infty} \left[\frac{\| u_q^{(k)} - u_q \|_{E(\Omega)}}{\| u_q^{(0)} - u_q \|_{E(\Omega)}} \right]^{1/k} \right\}$$

is a better measure for the actual performance of the HCM.

Remark 3.1. As we will see in the following section, we always have $\rho \leq \kappa$. In our experience the observed rate of convergence is much closer to ρ than κ except for possibly the very few first steps. $\qquad \square$

Remark 3.2. Algorithm 3.1 could be interpreted in the usual fashion as a multiplicative subspace correction. Since it is based on the hierarchy of models, we shall refer to it as *Hierarchic Subspace Correction Method* (HCM for short).

Along the same lines, of course, also additive subspace corrections based on the splitting (3.1) could be considered. $\qquad \square$

4. FOURIER CONVERGENCE ANALYSIS OF THE HCM - I

In this section we give a convergence proof for the HCM using Fourier analysis in the case of problem (2.5). To this end we define for $\alpha \in \mathbb{R}_+^n$ the (hyper) rectangle

(4.1)
$$\hat{\omega}_\alpha := \left\{ x \in \mathbf{R}^n \mid -\frac{\alpha_i}{2} < x_i < \frac{3\alpha_i}{2} \right\}.$$

A function w on $\mathbf{R}^n$ is called α-*periodic*, if

$$w(x + \alpha_i e_i) = w(x), \quad \text{a.e.} \quad x \in \mathbf{R}^n, \quad \forall i.$$

The vector $\beta = (\beta_1, ..., \beta_n)$ corresponding to α is defined by

$$\alpha = \left(\frac{\pi}{\beta_1}, \frac{\pi}{\beta_2}, ..., \frac{\pi}{\beta_n} \right).$$

To present the weak form of (2.5) in the periodic setting, we collect first a few notions on periodic function spaces (for more on the asymptotic analysis of hierarchical plate models in the periodic setting, we refer to [15]). Throughout the present section we will understand all Sobolev spaces as spaces of complex valued functions and indicate by a bar the conjugate complex quantity. By $H^1_{per}(\hat{\omega}_\alpha) \subset H^1(\hat{\omega}_\alpha)$ we denote the space of all 2α-periodic functions $w \in H^1(\hat{\omega}_\alpha)$.

If $w \in H^1_{per}(\hat{\omega}_\alpha)$, then let

(4.2)
$$w(x) = (2\pi)^{-n} \sum_{m \in Z^n} w_m e^{-i\langle x, [m\beta]\rangle}$$

where the Fourier coefficients w_m are given by

$$w_m = (\beta_1 \beta_2 ... \beta_n) \int_{x \in \hat{\omega}} w(x) e^{i\langle x, [m\beta]\rangle} dx$$

and $[m\beta]$ denotes $(m_1\beta_1, m_2\beta_2, ..., m_n\beta_n)$. For $v(x) = (2\pi)^{-n} \sum_{m \in Z^n} v_m e^{i\langle x, [m\beta]\rangle}$ we also have *Parseval's formula*

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$$(4.3) \qquad \int_{x \in \hat{\omega}_*} w \bar{v} \, dx \;=\; (2\pi)^{-n} (\beta_1 \beta_2 \dots \beta_n)^{-1} \sum_{\mathbf{m} \in Z^*} w_{\mathbf{m}} \bar{v}_{\mathbf{m}}.$$

Fourier transforms of vector functions are taken componentwise. Analogously, by $H^1_{per}(\hat{\Omega}_\alpha)$ we denote the set of all $w(x,y) \in H^1(\hat{\Omega}_\alpha)$ which are α-periodic with respect to the first variable in

$$(x,y) \in \Omega_\alpha := \hat{\omega}_\alpha \times (-d,d).$$

From Parseval's equation we obtain immediately

Proposition 4.1. *For all* $u \in per^1(\hat{\Omega}_\alpha)$

$$(4.4) \quad \| u \|^2_{E(\hat{\Omega}_\alpha)} = \int_{\hat{\omega}_\alpha} \int_{-d}^{d} \left\{ a\left[\frac{y}{d}\right] \left[\frac{\partial u}{\partial y}\right]^2 + b\left[\frac{y}{d}\right] (\nabla_x u)^T C \nabla_x u \right\} dx \, dy$$

$$= (2\pi)^{-1} (\beta_1 \dots \beta_n)^{-1} \sum_{\mathbf{m} \in Z^*} \int_{-1}^{1} \left\{ d^{-1} a(z) |u'_{\mathbf{m}}(z)|^2 + db(z)[\mathbf{m}\beta]^T C[\mathbf{m}\beta] |u_{\mathbf{m}}(z)|^2 \right\}$$

The expression (4.4) in Proposition 4.1 is a norm on the subspace

$$\bar{H}^1_{per}(\hat{\Omega}_\alpha) \;:=\; \bar{H}^1(\hat{\Omega}_\alpha) \cap \left\{ u \mid \int_{\Omega_\alpha} u \, dx \, dy = 0 \right\}.$$

Further define

$$\bar{L}^2(\hat{\Omega}_\alpha) \;:=\; L^2_{per}(\hat{\omega}_\alpha) \cap \left\{ u \mid \int_{\hat{\omega}_\alpha} u \, dx = 0 \right\}.$$

Defining $S_{per}(q) := H^1_{per}(\hat{\omega}_\alpha) \otimes V(q)$ and $\bar{S}_{per}(q) := \bar{H}^1_{per}(\hat{\omega}_\alpha) \cap S_{per}(q)$, we can write $u_q(x,y) = X(x)^T \left[\varphi\left[\frac{y}{d}\right] \right] \in S_{per}(q)$ in the form

$$(4.5) \qquad u_q \;=\; (2\pi)^{-n} \sum_{\mathbf{m} \in Z^*} \chi_{\mathbf{m}}^T \varphi\left[\frac{y}{d}\right] e^{-i<x_*[\mathbf{m}\beta]>}$$

where $\chi_{\mathbf{m}} = (\chi_{m,0}, \dots, \chi_{m,q})$ are (the vectors of) Fourier coefficients of order m of the

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(vector) function $X(x)$. Moreover

$$\|u_q\|_{E(\hat{\Omega}_a)}^2 = (2\pi)^{-n}(\beta_1...\beta_n)^1 d^{-1} \sum_{\mathbf{m} \in \mathbb{Z}^n} \bar{\chi}_{\mathbf{m}}^T \{d^2[\mathbf{m}\beta]^T C[\mathbf{m}\beta]A + B\}\chi_{\mathbf{m}}$$

(4.6)

$$= (2\pi)^{-n}(\beta_1...\beta_n)^{-1} d^{-1} \sum_{\mathbf{m} \in \mathbb{Z}^n} \bar{\chi}_{\mathbf{m}}^T L(d\eta_{\mathbf{m}})\chi_{\mathbf{m}},$$

(4.7)
$$\eta_{\mathbf{m}} := ([\mathbf{m}\beta]^T C[\mathbf{m}\beta])^{1/2}$$

and where the matrices A and B are given in (2.19). Then the symbol of the differential

operator $L(D_x)$ in (2.18) is

(4.8)
$$L(\eta_{\mathbf{m}}) := \eta_{\mathbf{m}}^2 A + B$$

Due to Proposition 4.1. the norm defined by (4.6) is a norm on the subspace

$\mathcal{S}_{per}(q)$.

<u>Remark 4.1.</u> If $\varphi_0 = \text{const}$ and $\int_{-1}^1 \varphi_j dz = 0$ for $j \geq 1$, we find that

$$u_q \in \mathcal{S}_{per}(q) \Leftrightarrow \chi_{00} = 0 \text{ in } (4.5).$$

More generally, for $\{\varphi_j\}_0^q$, let $1 = e^T \varphi$ for some $e \in \mathbb{R}^{q+1}$. Then

$$u_q \in \mathcal{S}_{per}(q) \Leftrightarrow e^T \chi_{\mathbf{0}} = 0 \text{ in } (4.5). \qquad \square$$

By $B_a(\cdot, \cdot)$ and $F_a(\cdot)$ we denote the forms in (2.9), (2.10) with integrations

over Ω_a and ω_a, respectively (and using $\bar{v}$ instead of v).

<u>Proposition 4.2.</u> *Let* $f \in L^2(\hat{\omega}_a)$, $f(x) = (2\pi)^{-n} \sum_{\mathbf{m} \in \mathbb{Z}^n} f_{\mathbf{m}} e^{-i<x,[\mathbf{m}\beta]>}$. *Then*

$\int_{\hat{\omega}_a} f(x)dx = 0 \Leftrightarrow f_0 = 0$ *and if*

(4.9) $$f_0 = 0$$

then the boundary value problem : Find $u \in \mathring{H}^1_{per}(\hat{\Omega}_\alpha)$ *such that*

(4.10) $$B_\alpha(u,v) = F_\alpha(v), \quad \forall v \in \mathring{H}^1_{per}(\hat{\Omega}_\alpha)$$

and the reduced problem: Find $u_q \in S_{per}(q)$ *such that*

(4.11) $$B_\alpha(u_q,v) = F_\alpha(v), \quad \forall v \in S_{per}(q)$$

both admit solutions which are unique up to constants. □

We shall now link the problem (2.5) on Ω to an equivalent one on $\hat{\Omega}_\alpha$. This will be done by an extension operator $\mathscr{E}$ defined as follows. We subdivide $\hat{\omega}_\alpha$ into 2^n hypercubes and identify one of them with ω_α in (2.1). We extend $u \in H^1(\Omega,\Gamma)$ to $\bar{u} \in \mathring{H}^1(\hat{\Omega}_\alpha)$ as follows: in 2^{n-1} octants we select $\bar{u} = -u$, with the argument of u properly shifted, and in the remaining $2^{n-1} - 1$ octants we select $\bar{u} = u$. (We note that $u = 0$ on $\partial\omega_\alpha \times (-d,d)$ and that this guarantees $\bar{u} \in H^1_{per}(\hat{\Omega}_\alpha)$.) In the same way we extend the function f. Here, of course, only $\tilde{f} \in L^2(\hat{\omega}_\alpha)$.

Proposition 4.3. *The extension* $\mathscr{E} : H^1(\Omega,\Gamma) \to \mathring{H}^1_{per}(\hat{\Omega}_\alpha)$ *is continuous and isometric, in the sense that*

(4.12) $$\|\tilde{u}\|^2_{E(\hat{\Omega}_\alpha)} = 2^n \|u\|^2_{E(\Omega)}, \quad \|\tilde{f}\|^2_{L^2(\hat{\omega}_\alpha)} = 2^n \|f\|^2_{L^2(\omega)}.$$

Proof: Due to the homogeneous essential boundary conditions satisfied by u on Γ we observe that $\tilde{u} \in H^1(\hat{\Omega}_\alpha)$ and that $\tilde{u} = 0$ on $\partial\hat{\omega}_\alpha \times (-d,d)$. Hence it admits an

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α-periodic extension to $\mathbb{R}^n \times (-d,d)$ which is locally in H^1. This implies that $\tilde{u} \in H^1_{per}(\hat{\Omega}_\alpha)$. The properties $\int_{\hat{\Omega}_\alpha} \tilde{u}\, dxdy = 0$ and (4.12) are obvious from the definition of $\mathscr{E}$. $\qquad\qquad \square$

Remark 4.2. We have actually shown the extension $\mathscr{E}$ to be continuous from $H^1(\Omega,\Gamma)$ onto $\mathring{H}^1_{per}(\hat{\Omega}_\alpha)$, the subspace of $H^1_{per}(\hat{\Omega}_\alpha)$ of functions the traces of which vanish on the hyperplanes $x_i = \alpha_i/2$, $i = 1,2,...,n$. $\qquad\qquad \square$

Remark 4.3. The space $\mathring{H}^1_{per}(\hat{\Omega}_\alpha)$ is a closed, linear subspace of $H^1_{per}(\hat{\Omega}_\alpha)$ and we define

$$\mathring{S}_{per}(q) := \mathring{S}_{per}(q) \cap \mathring{H}^1_{per}(\hat{\Omega}_\alpha).$$

Then, if we set for given $f \in L^2(\omega_\alpha)$

$$F_\alpha(v) := \int_{\hat{\omega}_\alpha} \tilde{f}(x)(v(x,d) + v(x,-d))dx,$$

(4.10) and (4.11) admit unique solutions $\hat{u} \in \mathring{H}^1_{per}(\hat{\Omega}_\alpha)$, $\hat{u}_q \in \mathring{S}_{per}(q)$. Their restrictions to Ω coincide with the solutions of (2.8) and (2.17), respectively. $\qquad \square$

Remark 4.2 is the basis for an *equivalent* formulation of the HCM Algorithm 3.1 in the periodic setting. To this end we introduce, analogous to (3.1), the subspace decomposition

(4.13) $$\mathring{S}_{per}(q) = \mathring{S}_0 + \mathring{S}_1 + \cdots + \mathring{S}_q$$

and we observe that $\mathscr{E}: S_j \longmapsto \mathring{S}_j$ continuously. Then the periodic version of Algorithm 3.1 is defined exactly as in (3.4) and, due to Remark 4.3, the restrictions of the resulting iterates $\hat{u}_q^{(n)} \in \mathring{S}_{per}(q)$ to ω_α are those produced by Algorithm 3.1, i.e.

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(4.14)
$$\hat{u}_q^{(n)}|_{\omega_a} = u_q^{(n)}, \quad \forall n.$$

From this equivalent periodic formulation of Algorithm 3.1, however, a precise estimate

of the convergence factors ρ and κ in (3.5) and (3.6) can be derived.

__Theorem 4.1__. *Let* $u_q^{(k)}(x,y) \in \mathcal{S}_{per}(q)$ *denote the k-th iterate of the successive subspace*

correction algorithm 3.1. Let further for a relaxation parameter $0 < \theta < 2$

(4.15)
$$M(\theta,\eta) := (D + \theta E)^{-1}((1 - \theta D - \theta E^H)$$

be the SOR iteration matrix corresponding to $L(\zeta)$ *defined in (4.8) (as it is usual,*

$D(\zeta)$ *and* $E(\zeta)$ *denote the diagonal and strictly lower triangular parts, respectively, of*

$L(\zeta)$). *Then*

    i)   *the contraction constant* κ *in (3.5) admits the estimate*

(4.16)
$$\kappa^2 \leq \sup_{\mathbf{m} \in Z_o^a} \{\lambda(\theta, \eta_\mathbf{m} d)\}$$

where $\lambda(\theta,\eta)$ *is the largest eigenvalue of*

$$M(\theta,\eta)^H L(\eta) M(\theta,\eta) x = \lambda(\theta,\eta) L(\eta) x,$$

and

$$Z_o^a = Z^a \cap \{\mathbf{m} | e^{-i<x,(\mathbf{m}\beta)>} \in \hat{H}_{per}^1(\hat{\omega}_a)\}$$

    ii)   *the asymptotic convergence rate* ρ *in (3.6) admits the estimate*

(4.17)
$$\rho = \sup_{\mathbf{m} \in Z_o^a} \{r(M(\theta, \eta_\mathbf{m} d))\}$$

where $r(.)$ *denotes the spectral radius.*

__Proof:__ We proceed in several steps.

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i) We write $\hat{u}_q(x,y) = X(x)^T\varphi\left[\dfrac{y}{d}\right]$, $\quad \hat{u}_q^{(k)}(x,y) = X^{(k)}(x)^T\varphi\left[\dfrac{y}{d}\right]$

and have the Fourier expansions

$$X(x) = (2\pi)^{-n)} \sum_{m\in Z_q^n} X_m\, e^{-i<x,[m,\beta]>},$$

where

$$X^{(k)}(x) = (2\pi)^{(-n)} \sum_{m\in Z_q^n} X_m^{(k)} e^{-i<x,[m,\beta]>}$$

$$X_m^{(k)} = \left(\chi_{m,0}^{(k)}, \chi_{m,1}^{(k)},\ldots,\chi_{m,q}^{(k)}\right)^T, \quad \text{etc.}$$

Let further

$$\tilde{f}(x) = \mathcal{F}f = (2\pi)^{-n} \sum_{m\in Z_q^n} f_m e^{-i<x,[m,\beta]>}.$$

Then the Fourier coefficients χ_m of the vector function $\chi(x)$ are determined from the linear system

(4.18)
$$L(\eta_m d)\chi_m = 2df_m\varphi(1), \quad m \in Z_0^n$$

where

$$\eta_m := ([m\beta]^T C[m\beta])^{1/2}.$$

The crucial observation is now that the successive subspace correction algorithm 3.1 becomes, in our periodic setting, a (frequency dependent) SOR-method for the solution of the hermitean linear systems (4.18), i.e.

$$\chi_m^{(k+1)} = M(\theta,\eta_m d)\chi_m(k) + 2df_m\tilde{M}(\theta,\eta_m d)\varphi(1),$$

where

(4.19)
$$\tilde{M}(\theta,\eta_m d) = \theta(D(\eta_m d) + \theta E(\eta_m d))^{-1}.$$

In particular, no Fourier modes that are absent in the right hand side $\tilde{f}(x)$ and the

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initial approximation $X^{(0)}(x)$ will be excited in the course of the iteration.

ii)   From (4.10) we obtain that

$$\|u_q - u_q^{(k)}\|^2_{E(\Omega_c)} = 2^{-a} \|\hat{u}_q - \hat{u}_q^{(k)}\|^2_{E(\hat{\Omega}_c)}$$

$$= 2^{-a} C_0 \sum_{m \in Z_i} \left(x_m - x_m^{(k)}\right)^H L(\eta_m d)\left(x_m - x_m^{(k)}\right)$$

where $C_0 = d^{-1}(2\pi)^{-a}(\beta_1 ... \beta_a)^{-1}$.

With the Fourier coefficients of the error $X - X^{(k)}$,

$$Y_m := x_m - x_m^{(k)}, \quad m \in Z_0^a,$$

we find

$$\|\hat{u}_q - \hat{u}_q^{(k)}\|^2_{E(\hat{\Omega}_c)} = C_0 \sum_{m \in Z_i} \|Y_m^{(k)}\|^2_E$$

where

$$\|Y_m^{(k)}\|^2_E = Y_m^{(k)H} L(\eta_m d) Y_m^{(k)}$$

$$= Y_m^{(k-1)H} M(\theta, \eta_m d)^H L(\eta_m d) M(\theta, \eta_m d) Y_m^{(k-1)}$$

$$\leq \lambda(\theta, \eta_m d) \|Y_m^{(k-1)}\|^2_E .$$

This proves (4.16).

iii) To prove (4.17), we first establish a perturbation result. Recall that

$L = \eta^2 A + B$,   A and B as in (2.19) and (4.8). Then

$$D = \eta^2 D_0 + D_1, \quad E = \eta^2 E_0 + E_1,$$

and we have for sufficiently large η

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$$M = ((\eta^2 D_0 + D_1) + \theta(\eta^2 E_0 + E_1))^{-1} \left\{ (1-\theta)(\eta^2 D_0 + D_1) - \theta(\eta^2 E_0 + E_1)^H \right\}$$

$$= \left((D_0 + \theta E_0) + \eta^{-2}(D_1 + \theta E_1) \right)^{-1} \cdot \left\{ \left((1-\theta)D_0 - \theta E_0^H \right) + \eta^{-2} \left((1-\theta)D_1 - \theta E_1^H \right) \right\}$$

$$= \left(I + \eta^{-2}(D_0 + \theta E_0)^{-1}(D_1 + \theta E_1) \right)^{-1} (D_0 + \theta E_0)^{-1} \left\{ \left((1-\theta)D_0 - \theta E_0^H \right) + \eta^{-2} \left((1-\theta)D_1 - \theta E_1^H \right) \right\}$$

$$= (D_0 + \theta E_0)^{-1} \left((1-\theta)D_0 - \theta E_0^H \right) + R$$

where the remainder R is given explicitly by

$$\left\{ \sum_{j=1}^{\infty} \left(\cdot \eta^{-2}(D_0 + \theta E_0)^{-1}(D_1 + \theta E_1) \right)^j \right\} (D_0 + \theta E_0)^{-1} \left\{ \left((1-\theta)D_0 - \theta E_0^H \right) + \eta^{-2} \left((1-\theta)D_1 - \theta E_1^H \right) \right\}$$

$$+ \eta^{-2}(D_0 + \theta E_0)^{-1} \left((1-\theta)D_1 - \theta E_1^H \right)$$

provided that $\eta > \eta_0$, η_0 is selected sufficiently large to ensure the convergence of

the Neumann series. Hence we have shown that, as $\eta \to \infty$,

(4.20)
$$M(\theta,\eta) = M_A(\theta) + R(\theta,\eta)$$

where $M_A(\theta)$ is the (η – independent) SOR matrix corresponding to A and

$\|R\| = 0(\eta^{-2})$ as $\eta \to \infty$ in any fixed, η-independent matrix norm.

    iv) We prove (4.17). As before, we find for $k \in N_0$

$$\| u_q - u_q^{(k)} \|_E^2 = 2^{-n} C_0 \sum_{m \in Z_q} \left(Y_m^{(0)} \right)^H (M^k)^H L(\eta_m d) M^k Y_m^{(0)}$$

$$= 2^{-n} C_0 \sum_{m \in Z_q} \| M^k Y_m^{(0)} \|_{L^m}^2$$

where $\| Y \|_{L^m}^2 := \| Y^H L Y \|_2^2$.

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As it is well known, for every matrix M and every $\epsilon > 0$ there exists a vector norm $\left|\left|\left| \cdot \right|\right|\right|_M$ on C^{q+1} such that for the induced l.u.b matrix norm (which we again denote by $\left|\left|\left| \cdot \right|\right|\right|_M$) there holds

(4.21)
$$\left|\left|\left| M \right|\right|\right| \leq r(M) + \epsilon/2, \quad \epsilon > 0$$

(the norm $\left|\left|\left| \cdot \right|\right|\right|_M$ depends on ϵ).

In particular, we observe that there exists a constant $c \geq 1$ independent of η) such that

(4.22a)
$$\left|\left|\left| R(\theta,\eta) \right|\right|\right| \leq c/\eta^2, \quad \eta \to \infty.$$

(4.22b)
$$r(M_A) - c/\eta^2 \leq r(M) \leq r(M_A) + c/\eta^2.$$

Now we fix $\epsilon > 0$ and introduce the index sets

$$\mathfrak{C}_- = \left\{ m \in Z_0^a \mid c/(\eta_m^2 d^2) < \epsilon/4 \right\}, \quad \mathfrak{C}_+ = Z_0^a \backslash \mathfrak{C}_-$$

with the constant c as in (4.22) and $\eta_m d > \eta_0$. Then we split

$$\left\| u_q - u_q^{(k)} \right\|_E^2 = 2^{-a} C_0 \left\{ \sum_{m \in \mathfrak{C}_-} + \sum_{m \in \mathfrak{C}_+} \right\}$$

and discuss both sums separately. For $m \in \mathfrak{C}_-^d$, we have

$$\left\| M^k Y_m^{(0)} \right\|_{L^M} = \left(Y_m^{(0)} \right)^H (M^k)^H L(\eta_m d) M^k Y_m^{(0)}$$

$$= \left(Y_m^{(0)} \right)^H (M^k)^H \left(\eta_m^2 d^2 A + B \right) (M^k) Y_m^{(0)}$$

$$\leq \eta_m^2 d^2 \left\| A^{1/2} M^k Y_m^{(0)} \right\|_2^2 + \left(Y_m^{(0)} \right)^H (M^k)^H B M^k Y_m^{(0)}.$$

We estimate the first term by

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$$\eta_m^2 d^2 \| A^{1/2} M^k Y_{m^*} \|_2^2 = \eta_m^2 d^2 \| A^{1/2} (M_A + R)^k Y_m^{(0)} \|_2^2$$

$$\leq \eta_m^2 d^2 \| A \|_2 \Lambda_1(\epsilon, d) \| \| (M_A + R)^k \| \|_{M_A}^2 \| Y_m^{(0)} \|_2^2$$

$$\leq \eta_m^2 d^2 \Lambda_1^2(\epsilon, d) \operatorname{cond}_2(A) (\| \| M_A \| \|_{M_A} + \| \| R \| \|_{M_A})^{2k} \| Y_m^{(0)} \|_{L^\mu}^2$$

where $\Lambda_1(\epsilon, d)$ is the equivalence constant between $\| \cdot \|_2$ and $\| \| \cdot \| \|_{M_A}$ for

sufficiently large k. For sufficiently large k, the second term can be estimated by

$$\left(Y_m^{(0)} \right)^H (M^k)^H B M^k Y_m^{(0)} \leq r(B) \| M^k Y_m^{(0)} \|_2^2 \Lambda_1^2(\epsilon, d)$$

$$\leq \eta_m^2 d^2 \epsilon / 4 \cdot r(B) \| A^{1/2} M^k Y_m^{(0)} \|_2^2 \Lambda_{min}^{-1}(A) \Lambda_1^2(\epsilon, d)$$

$$= \Lambda_2(\epsilon, d) \eta_m^2 d^2 \| A^{1/2} M^k Y_m^{(0)} \|_2^2.$$

Then we continue as in the estimate of the first term. By (4.21), this yields with a new

constant $\Lambda(\epsilon, d)$ the estimate

$$\| M^k Y_m^{(0)} \|_{L^\mu}^2 \leq \Lambda(\epsilon, d) \operatorname{cond}_2(A) (r(M_A) + \epsilon/4 + \epsilon/4)^{2k} \| Y_m^{(0)} \|_{L^\mu}^2,$$

and hence we obtain for every $\epsilon > 0$, $0 < d < 1$ and $k \in N_0$

$$(4.23) \qquad \sum_{m \in \mathfrak{C}_-} \| M^k Y_m^{(0)} \|_{L^\mu}^2 \leq \operatorname{cond}_2(A) \Lambda(\epsilon, d) (r(M_A) + \epsilon/2)^k \sum_{m \in \mathfrak{C}_-} \| Y_m^{(0)} \|_{L^\mu}^2$$

We consider the case $m \in \mathfrak{C}_+$. Since $\mathfrak{C}_+$ is a bounded set for fixed ϵ and d,

we estimate

$$\sum_{\mathbf{m}\in\mathfrak{C}_+} \|M^k Y_\mathbf{m}^{(0)}\|_{L^\mu}^2 \ \le\ \sum_{\mathbf{m}\in\mathfrak{C}_+} \|M^k\|_{L^\mu}^2 \|Y_\mathbf{m}^{(0)}\|_{L^\mu}^2$$

(4.24)
$$\le\ \sum_{\mathbf{m}\in\mathfrak{C}_+} \Lambda_\mathbf{m}(\epsilon,d)\, |||M^k|||_M^2 \|Y_\mathbf{m}^{(0)}\|_{L^\mu}^2$$

$$\le\ \left(\sup_{m\in\mathfrak{C}_+} \Lambda_\mathbf{m}(\epsilon,d) \right) \sum_{\mathbf{m}\in\mathfrak{C}_+} (r(M(\theta,\eta_\mathbf{m}d)) + \epsilon)^{2k} \|Y\|_{L^\mu}^2 .$$

From (4.22b), we see that

$$r(M_A(\theta)) \ \le\ r(M(\theta,\eta_\mathbf{m}d)) + \epsilon/2, \quad \forall \mathbf{m}\in\mathfrak{C}_-$$

so we can combine (4.23) and (4.24) into

$$\|u_q - u_q^{(k)}\|_E^2 \ =\ 2^{-n}C_1(\epsilon,d) \sum_{\mathbf{m}\in Z_0^n} (r(M(\theta,\eta_\mathbf{m}d)) + \epsilon)^{2k}$$

$$\le\ 2^{-n}(C_0 C_1(\epsilon,d)(\bar{r}+\epsilon)^{2k} \sum_{\mathbf{m}\in Z_0^n} \|Y_\mathbf{m}^{(0)}\|_{1/2}^2$$

$$=\ C_1(\epsilon,d)(\bar{r}+\epsilon)^{2k} \|u_q - u_q^{(0)}\|_E^2 ,$$

where

(4.25)
$$\bar{r}(\theta) \ =\ \sup_{\mathbf{m}\in Z_0^n} r(M(\theta,\eta_\mathbf{m}d))$$

and C_1 depends on ϵ and d, but is independent of k. Hence

$$\lim_{k\to\infty} \left[\frac{\|u_q - u_q^{(k)}\|_E}{\|u_q - u_q^{(0)}\|_E} \right]^{1/k} \ \le\ (\bar{r}+\epsilon) \overline{\lim_{k\to\infty}} \, (C_1(\epsilon,d))^{1/k}$$

$$=\ \bar{r} + \epsilon.$$

Since $\epsilon > 0$ and $0 < d < 1$ were arbitrary, the assertion (4.17) follows. $\square$

Remark 4.3. In (4.17) we have taken the sup over all $\eta_\mathbf{m}d \in Z_0^m$. Obviously we get

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an upper bound when we define

$$\bar{r}(\theta) \;=\; \sup_{\eta > 0} \{r(M(\theta,\eta))\}.$$

Theorem 4.1 has several important consequences which we discuss next. The crucial observation was (4.16) and (4.17) which states that in the periodic setting the HCM becomes simply a (parameter dependent) SOR method for the solution of the linear system (4.10). Since for every $m \neq 0$ the matrix $L(\eta_m d)$ is symmetric and positive definite, we have from Kahan's theorem (e.g., 12, Thm 8.3.5) the

Corollary 4.1. *There holds, for every* $d \in (0,1]$ *and* $m \in Z_0^n$,

$$(4.26) \qquad\qquad |\omega - 1| \;\leq\; \rho \;\leq\; \kappa \;\leq \sqrt{\lambda(\theta,\eta_m d)}\;.$$

Proof: We have Kahan's lower bound $|\omega - 1| \leq \rho(M(\theta,\eta))$ for all $\eta \neq 0$ (see, e.g. [12, Chap. 8]) and hence (4.26) follows from $\kappa \geq \rho(M(\theta,\eta))$. $\qquad\qquad \square$

Corollary 4.2. *The rate* κ *of convergence of the HCM is bounded from above independently of* d.

Proof: From Theorem 4.1 we have

$$(\kappa)^2 \;\leq\; \sup_{m \in Z_0^n} \{\lambda(\theta,\eta_m d)\}$$

$$\leq\; \sup_{m \in \mathbb{R}^+} \{\lambda(\theta,\eta)\}$$

where λ is as in (4.16).

The result follows since the matrices $M(\theta,\eta)$ and $L(\eta)$ are independent of d. $\qquad\qquad \square$

The SOR-iteration applied to a diagonal matrix L converges in one step.

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Consequently, if we can give a basis φ that renders A and B in (2.18) diagonal simultaneously, the matrix $L(\eta d)$ will become diagonal and the HCM will converge in one step.

Corollary 4.3. *Let* $A = \int_{-1}^{1} b(z)\psi\psi^T dz$, $B = \int_{-1}^{1} a(z)\psi'\psi'^T dz$ *with the functions* ψ *defined in (2.13)-(2.15). Let Q be the* $(q+1) \times (q+1)$ *matrix the columns of which are the eigenvectors of*

$$(4.27) \qquad\qquad Bx = \sigma_i^2 Ax$$

normalized so that $x^T Ax = 1$ *and* $0 \leq \sigma_i^2 \leq \sigma_{i+1}^2$. *Then the basis* $\varphi = Q^T\psi$ *renders* $L(\eta d)$ *in (4.14) diagonal and the HCM converges in one step, i.e.* $\kappa = 0$ *in (3.5).*

Proof. We have $x^T Bx = \sigma_i^2 x^T Ax = \sigma_i^2$ by (4.19). Now, due to $\varphi = Q^T\psi$,

$$\int_{-1}^{1} b(z)\varphi\varphi^T dz = Q^T AQ = 1,$$

$$\int_{-1}^{1} a(z)\varphi'\varphi'^T dz = Q^T BQ = \left\{0, \sigma_1^2, ..., \sigma_q^2\right\},$$

hence with $\sigma_0 = 0$

$$L(\zeta) = \zeta^2 A + B = \text{diag}\left\{\zeta^2 + \sigma_j^2\right\}_{j=0}^{q}. \qquad\qquad \square$$

Frequently, especially in elasticity problems, it is not possible to find a ζ-independent transformation Q which renders L diagonal. Nevertheless, as we shall see in the next section, it is often useful (and possible) to have $L(\zeta)$ approach diagonal form as $\zeta \rightarrow 0^+$ or as $\zeta \rightarrow \infty$.

Corollary 4.4. *Assume that* $0 < \theta < 2$. *Then*

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$1°$: $\lim\limits_{\eta \to 0} \lambda(\theta,\eta) = 0$ *if* $B = \int_{-1}^{1} a(z)\varphi'(\varphi')^T dz$ *is diagonal.*

$2°$: $\lim\limits_{\eta \to \infty} \lambda(\theta,\eta) = 0$ *if* $A = \int_{-1}^{1} b(z)\varphi\varphi^T dz$ *is diagonal.*

Proof. $1°$. Consider $L(\zeta)$ as $\zeta \to 0$. Since then $L(\zeta) = \zeta^2 A + B \to B$ and B is

diagonal, the assertion would follow if B were nonsingular. Since, however, rank $(B) =$

$q < \dim(B) = q + 1$, we assume that $\sigma_0^2 = 0$, $\varphi_0 =$ const. and that

$\int_{-1}^{1} b(z)\varphi_i\varphi_0 dz = 0$ for $i \geq 1$. This uncouples the first equation in (4.18) from the

remaining ones, i.e. the iteration matrix $M(\theta,\eta)$ attains block diagonal form:

$$M(\theta,\zeta) = \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}.$$

The corresponding lower right block of B is now diagonal and positive definite, hence

$1°$ follows.

$2°$. We write in this case

$$L(\zeta) = \zeta^2 (A + \zeta^{-2} B).$$

From the formula for M in Theorem 4.1, $2°$ we find that M corresponding to L in

(4.20) and M corresponding to $(A + \zeta^{-2} B)$ are identical (i.e. a scaling of L does not

alter M). Since A is diagonal and positive definite, and $\zeta^{-2} B \to 0$ as $\zeta \to \infty$, we

find $2°$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

In summary, we have shown that the influence of the basis $\{\varphi_j\}_{j=0}^{q}$ on the

convergence rate of the HCM Algorithms 3.1 can be analyzed quantitatively with the iteration matrix $M(\theta, \eta_m d)$ in (4.15). In the same vein any other iterative method that is based on the subspace decomposition (3.1) can be analyzed, in particular the parallel subspace correction method.

Remark 4.4. The analysis presented so far applied only to the case where ω is an interval or a rectangle and the boundary conditions are either homogeneous Dirichlet- or periodic conditions. However, analogous results can be obtained for arbitrary Lipschitz-domains $\omega \subset \mathbb{R}^n$, if the unknown coefficient functions $X(x)$ and their iterates $X^{(k)}(x)$ are expanded into series of eigenfunctions $\psi_k(x)$ of the operator $-\Delta_x$ in ω with appropriate homogeneous boundary conditions on $\partial\omega$.

5. FOURIER CONVERGENCE ANALYSIS OF THE HCM-II

In the previous section we introduced the tool of Fourier transformation of the reduced model (3.3) to obtain explicit estimates on the convergence rate κ in (3.5) and ρ in (3.6). This was achieved by taking the supremum overall wavenumbers m of the largest eigenvalue $\lambda(\theta, \eta_m d)$ of (4.16), resp. of $r(M(\theta, \eta_m d))$ of (4.17). Our purpose in the present section is to show that the eigenvalue curve $\lambda(\theta, \eta)$, resp. $r(M(\theta, \eta))$ contains more information: it is a factor by which the energy contained in error components of "wavelength" $\eta^{-1}d$ is reduced in one step of the HCM. We consider here the special case of (2.5) where

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(5.1) $a = b = 1, \quad C = 1$ (unit matrix),

so that $L = \Delta$ and $D_n = \dfrac{\partial}{\partial n}$. Then it is readily verified that the director functions ψ_j

in (2.13)-(2.15) are even polynomials of z and we list the first of them in the following

table.

TABLE 5.1. The function $\psi_j(z)$, $0 \le j \le 2$.

j	$\psi_j(z)$
0	1
1	$(3z^2 - 1)/6$
2	$(15z^4 - 30z^2 + z)/360$

The basis functions ψ_j which render A in (2.17) diagonal are orthogonal in

$L^2(-1,1)$ and we select them to be the even Legendre polynomials, i.e.

(5.2) $\psi_j(z) = L_{2j}(z)$

where $L_0(z) = 1$, $L_2(z) = (3z^2-1)/6$, etc.

Analogously, the functions which render B in (2.17) diagonal are orthogonal in

$H^1(-1,1)$ with respect to the inner product

(5.3) $(\varphi_i, \varphi_j) = \int_{-1}^{1} \varphi'_i \varphi'_j \, dz.$

A basis which satisfies $(\varphi_i, \varphi_j) = \delta_{ij}$ is therefore given by

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(5.4)
$$\varphi_0(z) = 1, \quad \varphi_j(z) = \sqrt{\frac{4j-1}{2}} \int_{-1}^{z} L_{2j-1}(\xi)d\xi, \quad j \geq 1$$

or, more explicitly, $\varphi_j(z) = (2(4j-1))^{-1/2}(L_{2j}(z) - L_{2j-2}(z))$.

Figures 5.1 and 5.2 depict the bounds $\rho(M(\theta,\eta))$ for $\theta = 1$ (i.e. no relaxation) over several orders of magnitude of the normalized frequency η for the basis function in (5.2) and (5.4), respectively. We remark that $L_0(z)$ and $L_2(z)$ are orthogonal in the $L^2(-1,1)$ and in the $H^1(-1,1)$ inner product (5.3) - hence we have $\lambda(\theta,\eta) = 0$ for $q = 1$ with the basis (5.2), in accordance with Corollary 4.3.

In accordance with Corollary 4.4, we have

(5.5)
$$\lim_{\eta \to \infty} \rho(M(1,\eta)) = 0 \quad \text{for (5.2)}$$

and

(5.6)
$$\lim_{\eta \to 0^+} \rho(M(1,\eta)) = 0 \quad \text{for (5.4)},$$

i.e. (5.2) is better for a rapid convergence on the high frequency components while (5.4) performs superior on the low frequencies. Of principal interest is the overall supremum $\rho(M(\theta,\eta))$ for $\theta=1$, $q = 1,2,3,4$ with the basis (5.4).

(5.7)
$$\sup_{\eta \in \mathbb{R}^+} \{\rho(M(1,\eta))\} =: \rho(q).$$

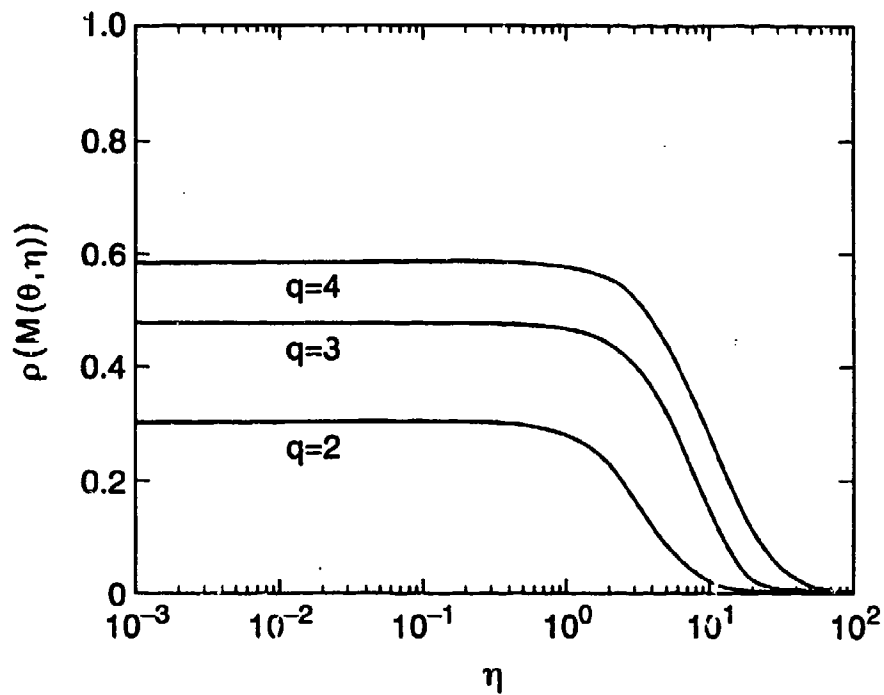In Table 5.2 we report the numerical values for the bases (5.2) and (5.4).

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Figure 5.1. $\rho(M(\theta,\eta))$ for $\theta=1$, $q = 2,3,4$ with the basis (5.2).
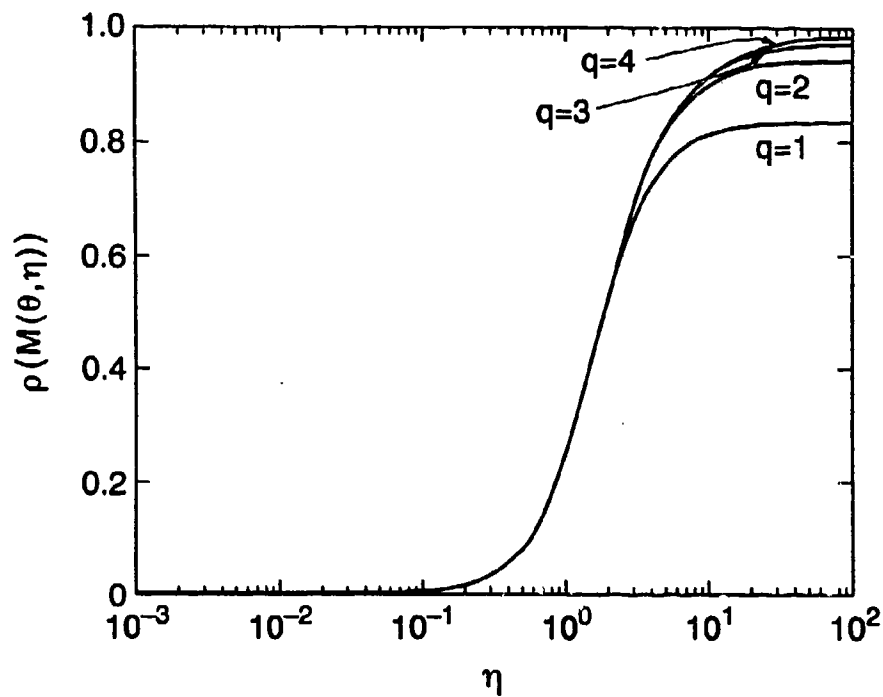

Figure 5.2. $\rho(M(\theta,\eta))$ for $\theta=1$, $q = 1,2,3,4$ with the basis (5.4).

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TABLE 5.2: $\rho(q)$ in (5.7) for the bases (5.2) and (5.4).

q	0	1	2	3	4
(5.2)	0	0	0.3000	0.4764	0.5833
(5.4)	0	0.833	0.9400	0.9694	0.9811

We see clearly that the basis (5.2) is more robust than (5.4), since it yields reasonably small contraction rates uniformly on all frequencies η . The performance of the basis (5.4) can be poor if the error contains high frequency components. In the following table we list for q = 1 and the basis (5.4), for small η .

TABLE 5.3. $\rho(q)$ for q = 1 and the basis (5.4).

η	0.001	0.01	0.1	0.25	0.5
$\rho(M(1,\eta))$	$0.33 \cdot 10^{-6}$	$0.33 \cdot 10^{-4}$	$0.33 \cdot 10^{-2}$	$0.22 \cdot 10^{-1}$	$0.75 \cdot 10^{-1}$

We see clearly that we have here

(5.8) $$\rho(M(1,\eta)) \sim \eta^2/3 \quad \text{as} \quad \eta \to 0^+.$$

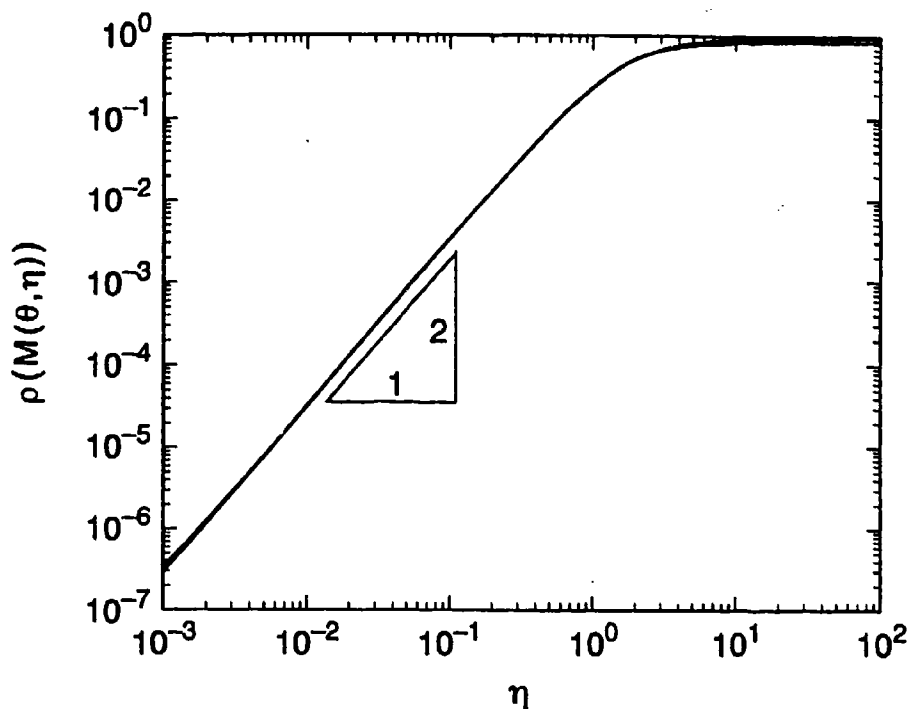In Figure 5.3 we depict this quantity also for higher q, with the slope according to (5.8).

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Figure 5.3. $\rho(M(\theta,\eta))$ for $\theta = 1$, $q = 1,2,3,4$ with basis (5.4), log-log scale.

ce further that (5.8) is practically independent of the model order for $\eta \leq 1$.

The above graphs show that we can expect generally a different performance of the HCM on high and low frequency components of the error $Y^{(k)}(x) := X(x) - X^{(k)}(x)$. This is of significance since the exact solution $X(x)$ consists, especially for small d, of two mutually distinct components: a smooth, asymptotic part and *boundary layers* $X_j^{BL}(x)$ of the form

$$(5.9) \qquad X_j^{BL}(x) = \{\chi_j^{BL} \exp(\sigma_j x/d)\}$$

where σ_j depends only on the model order q [16] and $\chi_j^{BL}(x)$ is smooth.

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In Table 5.4 we list the values of $\sigma_j(q)$.

TABLE 5.4: Boundary layer exponents $\sigma_j(q)$ for small q.

q	1	2	3	∞
$\sigma_1(q)$	$\sqrt{15}$	3.1529	3.1416	π
$\sigma_2(q)$		9.7498	6.4791	2π
$\sigma_3(q)$			18.0596	3π

It is further shown in [16] that

$$\sigma_j(q) \to j\pi \quad \text{as} \quad q \to \infty.$$

Comparing formally $\exp(-\sigma x/d)$ and $\exp([m\beta]x)$ with $\beta = \dfrac{\pi}{2}$ we expect large Fourier coefficients χ_m for $[m\beta] \sim \sigma/d$ or for

$$(5.10) \qquad\qquad \eta \;\sim\; \frac{\sigma\pi}{\alpha}.$$

We remark at this point that for certain compatible data $f(x)$ the components (5.9) vanish. Nevertheless, for arbitrary data the boundary layers (4.9) are generally present and we see from (5.10) that we can expect the contraction rate

$$(5.11) \qquad\qquad \rho \;\leq\; \kappa \;=\; \sqrt{\lambda\left[\theta, \frac{\sigma\pi}{\alpha}\right]}$$

for the HCM. If several distinct layers are present (cf. Table 5.4), we see that $\sigma\pi/\alpha$ can be quite large, hence, a decay of $\lambda(\theta,\eta)$ as $\eta \to \infty$ is highly desirable.

Remark 5.1. It follows from Theorem 4.1 only that $\lambda \leq \kappa$. Nevertheless, we found in all cases for the matrix M in (4.15) that $\rho = \lambda$.

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6. NUMERICAL EXPERIMENTS

In order to verify the applicability of our theory in a computational setting, we considered the following model problem:

$$(6.1) \qquad\qquad -\Delta u = 0 \quad \text{in} \quad (-1,1) \times (-d,d),$$

$$(6.2) \qquad\qquad u = 0 \quad \text{for} \quad x = \pm 1$$

$$(6.3) \qquad\qquad \frac{\partial u}{\partial y} = \pm 1 \quad \text{for} \quad y = \pm d.$$

The solutions exhibit boundary layers of the form (5.9) (see also [16]).

The problem (6.1)-(6.3) was discretized by a two dimensional FEM with rectangular elements $\omega_i \times (-d,d)$ where $\omega_i \subset (-1,1)$ are subintervals. Tensor product polynomial subspaces with degree p in horizontal and uniform degree q (only even shape functions were used here) in the transverse direction were used. We selected $p = 10$ to ensure that the models were solved accurately.

Remark 6.1. In practice a locally *variable degree* q is of high interest, see e.g. [13]. In this case the HCM becomes a block iterative method (here a block SOR) for the fully discrete system.

An exact solution was obtained by direct solution of the linear system and the iteration was stopped when

$$(6.4) \qquad\qquad \| x^{k} - x_{exact} \|_E \, / \, \| x_{exact} \|_E < 10^{-8}$$

where $\| \cdot \|_E$ denotes the (discrete) energy norm. We selected the initial guess $x^{(0)} = 0$ throughout (in practice when solving the higher order models, one should start from a converged solution of the lower order problem). As an averaged convergence rate we took

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the geometric mean of J iterates before convergence occurred, i.e.

$$(6.5) \qquad \mu = \left\{ \prod_{k^*+1-J}^{k^*} \frac{\| x^{(k)} - x_{exact} \|_E}{\| x^{(k-1)} - x_{exact} \|_E} \right\}^{1/J}$$

($J = \min\{10, k^*\}$ was selected). In Figure 6.1 we show the average convergence rates for models based on the director functions (5.2). The model orders $q = 0$ and $q = 1$ were not shown, since the HCM converges in one step here.

We observe in Figure 6.1 that the maximum convergence rates are very close to the values predicted by our Fourier analysis. Moreover, we observe that as $d \to 0$ all rates tend to 0.3, the theoretical value (cf. Fig. 5.1). This is due to the contribution of the $q = 3$ and 4 models being of higher order as $d \to 0$ and due to the 1-step convergence of HCM for $q = 1$.

Let us now turn to the HCM with the basis (5.4). Figure 6.2 shows the performance for (6.1)-(6.3).

Note that for the boundary layer case (Fig. 6.2), the convergence rates are once again close to the ones predicted in Table 5.2 which are in this case, for large d, very close to 1.

Finally we present in Figure 6.3 the convergence history for the HCM with (5.2) for various values of d and for $q = 2$. Figure 6.4 presents the analogous data for the basis (5.4).
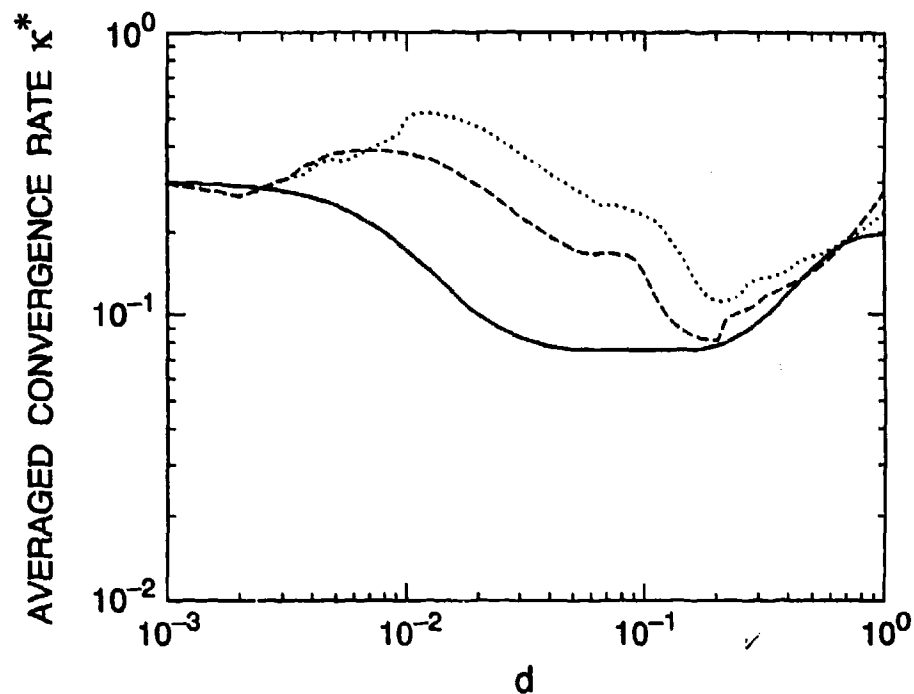
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Figure 6.1. Averaged convergence rate versus d with (5.2),
q = 2(—), 3(---), 4(· · ·).
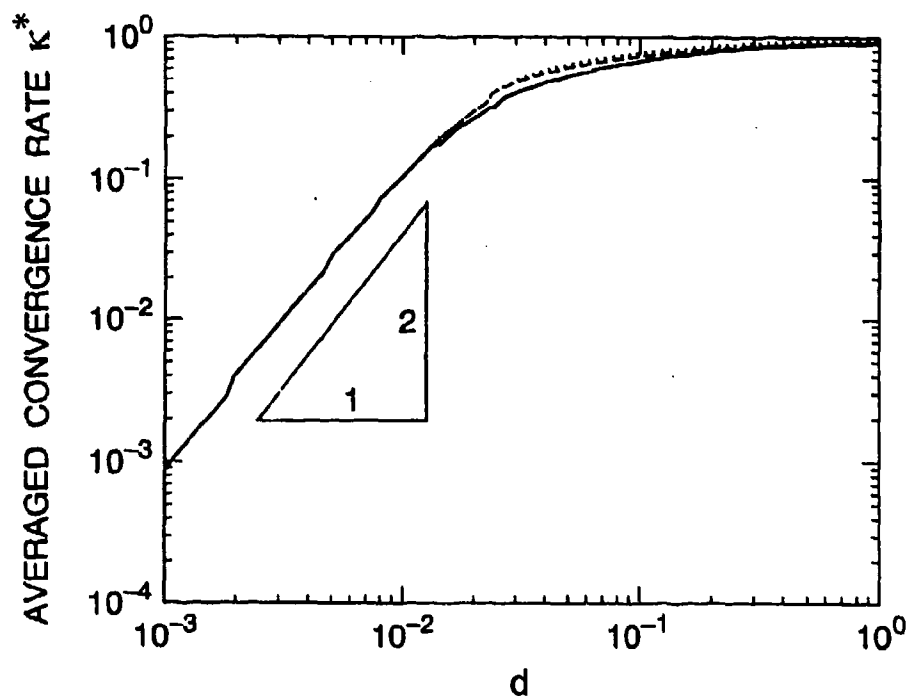

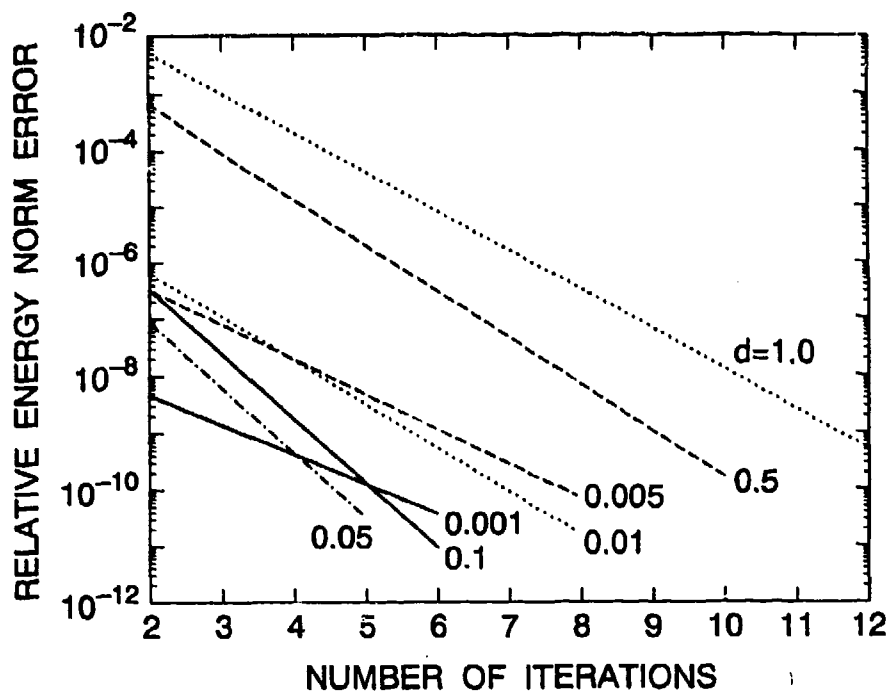Figure 6.2. Averaged convergence rate versus d with (5.4), q = 2(—), 3 (---), 4(· · ·).

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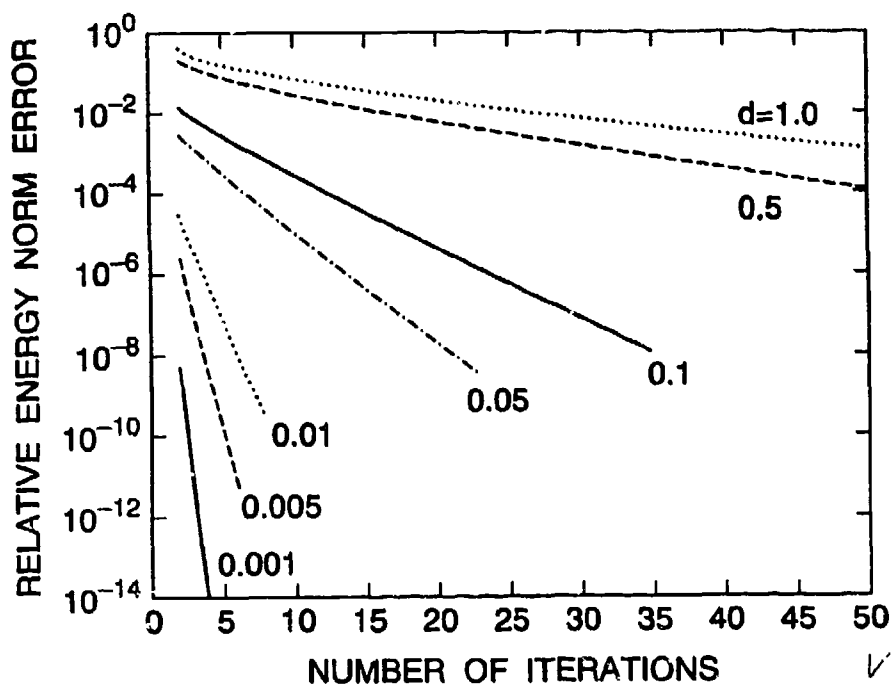Figure 6.3. Convergence history for q = 2, and basis (5.2).

Figure 6.4. Convergence history for q = 2, (5.4).

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7. CONCLUSIONS

We have shown that the hierarchic models can be solved very effectively by an iterative procedure, provided that the shape functions used (in y direction) are properly selected. The iterative procedure employed was based on a subspace decomposition with respect to the spectral order of the hierarchic plate model. Sharp estimates for the convergence rate of the subspace correction in dependence on the selection of the director functions were obtained using the Fourier transform. The technique will be extended to elasticity problems in the forthcoming second part of this work.

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